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BCK / BCH ALGEBRAS WITH MUTUALLY DISJOINT ELEMENTS

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ABSTRACT.

BCK/ BCH algebras are attractive topics for many researchers. Many results have been developed using BCK/ BCH algebras and their properties. In this paper, we have discussed some methods to construct BCK, BCH algebra with given finite sets having some specific conditions under some suitable binary operations defined therein. Some theorems and their proof along with examples have been provided using suitable binary operation.

KEYWORDS: BCK-Algebra, BCH-Algebra, Disjoint elements.

1. INTRODUCTION

BCK algebras were introduced in 1966, by K. Is´eki [4]. This notion originated from two different ways: (1) set theory, and (2) classical and non-classical propositional calculi. The BCK-operation * is an analogue of the set theoretical difference. The class of all BCK-algebras is a quasi-variety. Is´eki posed an interesting problem whether the class of BCK-algebras is a variety [2].

Now a days BCK-algebras have been studied by many authors and they have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory, topology, fuzzy set theory, and so on. Many researchers have developed numerous results using BCK/BCI/BCH algebras. This paper is based on the discussion on certain methods to construct BCK algebras. First, we will introduce the basic definitions and properties of BCK algebras.

2. LITERATURE REVIEW/EXPERIMENTAL DETAILS

A BCK-algebra is an important class of logical algebras introduced by K. Iséki [2] and was comprehensively examined by numerous scholars

Definition (2.1): - A BCK-Algebra is a system (X, *, 0) of the type (2,0), consisting of a non-empty set X, a binary operation * and a fixed element 0, satisfying the following postulates:

BCK (i) $0 \le x$
BCK (ii) $x \le x$
BCK (iii) $x \le y, y \le x \Rightarrow x = y$
BCK (iv) $x * (x * y) \le y$
BCK (v) $(x * y) * (x * z) \le (z * y)$

Here $x \le y$ is equivalent to $x * y = 0$.

Definition (2.2): - A BCK – algebra (X, *, 0) is said to be bounded if there exists an element $1 \in X$ such that $x \le 1$ for all $x \in X$. Here 1 is called the unit of X. In this case the complement of an element $x \in X$, denoted by Nx, is defined as $Nx = 1 * x$. [3]

Definition (2.3): - Let (X, *, 0) be a BCK – algebra.

(a) For $x, y \in X$, we defied $x \wedge y$ as follows:

$$x \wedge y = y * (y * x)$$

(b) An element $x \in X$ is said to be right (resp. left) disjoint from $y \in X$ w. r. t. the relation $\wedge$ if $x \wedge y = 0$ (resp. $y \wedge x = 0$).

(c) x is disjoint from y if it is both right and left disjoint from y. [6]

Definition (2.4): - Let (X, *, 0) be a BCK-Algebra and let $A \subseteq X$. Then A is called

(i) A sub algebra if x, y Є A ⇒ x * y Є A and

(ii) An ideal if for any x, y Є A, 0 Є A and x * y, y Є A ⇒ x Є B [1]

Definition (2.5): A BCH-algebra is an algebra (X, *, 0) of type (2, 0) satisfying the following conditions:

(i) $x * x = 0$,

(ii) $(x * y) * z = (x * z) * y$

(iii) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$

for all $x, y, z \in X$

Now, we mention the following facts.[8]

Proportion (2.1): (a) A BCK algebra having only two elements, both elements are mutually disjoint.

(b) Every subset of a BCK / BCI / BCC / BCH algebra having only mutually disjoint elements and the zero element is a sub algebra.

Now, we mention some results which are useful in the extension of BCK – algebra [7]

Theorem (2.1): Let (X, *, 0) be a BCK algebra where $X = \{0, a_1, a_2, \ldots, a_{n-1}, a_n\}$ and $S = \{0, a_1, \ldots\ldots., a_{n-1}\}$ is the set of mutually disjoint elements.

Then,

(a) $a_i * a_n \neq a_j$ where i, j = 1, 2, ……., n-1, $i \neq j$

(b) $a_i * a_n = a_n$ is true for at most one value of i and in this case $a_n * a_i = 0$ and $a_n * a_j = a_n$ for $i \neq j$

(c) if we take $a_k * a_n = a_n$ for a fixed value of k then $a_j * a_n = a_j$ or 0 for $j \neq k$. [6]

3. RESULT AND DISCUSSION

Now we prove the following result.

Theorem (3.1): Every set with finite number of elements can be made into a BCK – algebra under a suitable binary operation such that all elements of the set are mutually disjoint

Proof:

Let S be a finite set whose elements are indexed as $u_0, u_1, \ldots\ldots, u_{n-1}$. We take $u_0 = 0$ as zero element. Since all elements of S are mutually disjoint, is new of condition we define a binary operation * on S as follows:

$$\left.\begin{aligned} 0 * u_i = 0, u_i * 0 = u_i \\ u_i * u_j = u_i \quad for\ i \neq j \end{aligned}\right\} \ldots (3.1.1)$$

$$\left. u_j * u_i = u_j \quad for\ i \neq j \right\} \ldots (3.1.2)$$

where i, j = 1, 2, ……., n-1

From the above definition we see that conditions (i), (ii), (iii) of a BCK algebra is satisfied. We also see that

$(u_i * (u_i * u_i)) * u_i = 0$

and for $i \neq j$

$(u_i * (u_i * u_j)) * u_j = (u_i * u_i) * u_j = 0 * u_j = 0$

So, condition (iv) is satisfied.

To check conditions (v) we proceed as follows:

Let $i \neq j \neq k$. Then

$((u_i * u_j) * (u_i * u_k)) * (u_k * u_j) = (u_i * u_i) * u_k = 0 * u_k = 0.$

Let $i \neq j = k$. Then

$((u_i * u_j) * (u_i * u_k)) * (u_k * u_j) = (u_i * u_i) * u_o = 0$

Again, let $i = j \neq k$. Then

$((u_i * u_j) * (u_i * u_k)) * (u_k * u_i) = (0 * u_i) * u_k = 0 * u_k = 0.$

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This means that in all cases condition (v) is satisfied. Hence $(S, *, u_o = 0)$ is a BCK algebra such that all elements of S are mutually disjoint.
Now we discuss some examples

Example (3.1):

We consider systems {E, *, 0} and $\{E^1, *, 0\}$ where, E = {0, a, b, c} and E = {0, a, b, c, d} and binary operations are defined by tables

*	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Table (3.1)

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	b	0
c	c	c	c	0	0
d	d	c	b	a	0

Table (3.2)

Here table 3.1 represents a BCK – algebra in which the set S = {0, a, b} is a set of mutually disjoint elements and is a BCK subalgebra and the given BCK algebra is an extension of S.

In table 3.2 the set S = {0, a, b, c} is also a BCK algebra of disjoint elements but $(E^1, \bullet, 0)$ is not a BCK algebra because
$((b \bullet c) \bullet (b \bullet d)) \bullet (d \bullet c) = (b \bullet 0) \bullet a = b \bullet a = b \neq 0$

However, we have an extension in the following form

Theorem (3.2): Let (E, *, 0) be a finite BCK algebra such that elements $0 \equiv u_0, u_1, ..., u_{n-1}$ of E be mutually disjoint. Let b be an object not in E and let $E^1 = E \cup \{b\}$, we define

(a) a binary operation o in E^1 as follows:

$u_i \text{ o } u_j = u_i * u_j = u_i$ (3.1.3)
for i, j = 0, 1, 2, ……, n-1, i ≠ j

For i = j, $u_i \text{ o } u_i = u_i * u_i = 0$ (3.1.4)
0 o b = 0, b o b = 0, b o 0 = b (3.1.5)
$u_i \text{ o } b = u_i$, $b \text{ o } u_i = 0$ (3.1.6)
for i = 1, 2, ……., n-1
Then $(E^1, o, 0)$ is a BCK algebra.

(b) a binary operation ● in E^1 as follows:

$u_i \bullet u_j = u_i * u_j = u_i$; (3.1.7)
i, j = 0, 1, 2, ……, n-1
$0 \bullet b = b$, $b \bullet 0 = b$, $b \bullet b = 0$ (3.1.8)
$u_i \bullet b = u_i$, (3.1.9)
$b \bullet u_i = b$ (3.1.10)
for i = 1, 2, ……., n-1
Then $(E^1, \bullet, 0)$ is a BCH - algebra but not BCK – algebra

Proof:

(a) From the given information it follows that conditions (i), (ii) and (iii) of BCK-Algebra are Satisfied, Again
$(u_i \text{ o } (u_i \text{ o } b)) \text{ o } b = (u_i \text{ o } u_i) \text{ o } b = 0 \text{ o } b = 0$
and $(b \text{ o } (b \text{ o } u_i)) \text{ o } u_i = (b \text{ o } 0) \text{ o } u_i = b \text{ o } u_i = 0$
imply that condition (iv) of BCK-Algebra is satisfied.
Now we check condition (v) of BCK-Algebra for u_i, u_j, 0 and b.
We have

$((u_i \circ b) \circ (u_i \circ 0)) \circ (0 \circ b) = (u_i \circ u_i) \circ 0 = 0$

$((u_i \circ u_j) \circ (u_i \circ b)) \circ (b \circ u_j) = (u_i \circ u_i) \circ 0 = 0$

$((b \circ u_i) \circ (b \circ u_j)) \circ (u_j \circ u_i) = (0 \circ 0) \circ u_j = 0$

This means that all the conditions of BCK-Algebra are satisfied and hence $(E^1, o, 0)$ is a BCK algebra.

(b) we see that conditions (i) and (iii) are satisfied for the binary operation ● defined on E^1. To check conditions (ii) it suffices to check the condition for 0, b and u_i $(i \neq 0)$. We see that

for $i \neq j \neq k$, $(u_i \bullet b) \bullet u_j = u_i \bullet u_j = u_i$ and

$(u_i \bullet u_j) \bullet b = u_i \bullet b = u_i$,

$(0 \bullet b) \bullet u_i = b \bullet u_i = b$,

$(0 \bullet u_i) \bullet b = 0 \bullet b = b$,

$(u_i \bullet b) \bullet 0 = u_i \bullet 0 = u_i$,

$(u_i \bullet 0) \bullet b = u_i \bullet b = u_i$,

Hence $(E^1, \bullet, 0)$ is a BCH - algebra.

We take $0 \neq i \neq j$. Then

$((u_i \bullet u_j) \bullet (u_i \bullet b)) \bullet (b \bullet u_j) = (u_i \bullet u_i) \bullet b = 0 \bullet b = b \neq 0$.

So $(E^1, \bullet, 0)$ is not a BCK – algebra.

Now we discuss another method for extending the algebra E into a BCK – algebra E^1 by using theorem (3.1)

Theorem (3.3): We recall sets E and E^1 of theorem (3.2). We define a binary operation 0 and E^1 which satisfies equation (3.1.3) and (3.1.4).

Further, we assume that this binary operation also satisfies conditions

$u_k \odot b = b$ and $b \odot u_k = 0$ (3.1.11)

for a fixed suffix $k \neq 0$,

$u_i \odot b = u_i$ and $b \odot u_i = b$ (3.1.12)

for $i = 1, 2, \ldots\ldots, k-1, k+1, \ldots\ldots, n-1$.

Then $(E^1, \odot, 0)$ is a BCK – algebra.

Proof:

Most part of the proof follow from theorem (3.2) we need to check conditions (iv) and (v) for some particular elements of E^1. We have

$(u_k \odot (u_k \odot b)) \odot b = (u_k \odot b) \odot b = b \odot b = 0$

and $(b \odot (b \odot u_k)) \odot u_k = (b \odot 0) \odot u_k = b \odot u_k = 0$

Further, for $i \neq k$

$(u_i \odot (u_i \odot b)) \odot b = (u_i \odot u_i) \odot b = 0 \odot b = 0$

and $(b \odot (b \odot u_i)) \odot u_i = (b \odot b) \odot u_i = 0 \odot u_i = 0$

We also have

$((u_k \odot b) \odot (u_k \odot 0)) \odot (0 \odot b) = (b \odot u_k) \odot 0 = 0$

For $i \neq k$

$((u_i \odot u_k) \odot (u_i \odot b)) \odot (b \odot u_k) = (u_i \odot u_i) \odot 0 = 0$

and $((b \odot u_k) \odot (b \odot u_k)) \odot (u_k \odot u_i) = (b \odot 0) \odot u_k = 0$

Thus, we see that conditions (iv) and (v) are satisfied in all cases.

Hence $(E^1, \odot, 0)$ is a BCK – algebra.

Remark (3.1): If we replace condition (3.1.12) of the above theorem by $u_i \odot b = 0$ for some $i \neq k$ then $(E^1, \odot, 0)$ is not a BCK – algebra. However, it is a positive BCH – algebra

We choose $i \neq k$ for which $u_i \odot b = 0$. Then

$((u_i \odot u_k) \odot (u_i \odot b)) \odot (b \odot u_k) = (u_i \odot 0) \odot 0 = u_i \neq 0$

So, condition (v) of BCK-algebra is not satisfied. Now, we see that for such i

$(u_i \odot b) \odot u_k = 0 \odot u_k = 0$; $(u_i \odot u_k) \odot b = u_i \odot b = 0$

$(u_k \odot b) \odot u_i = b \odot u_i = b$; $(u_k \odot u_i) \odot b = u_k \odot b = b$

$(b \odot u_k) \odot u_i = 0 \odot u_i = 0$; $(b \odot u_i) \odot u_k = b \odot u_k = 0$

which means that condition (ii) is satisfied. So $(E^1, \odot, 0)$ is a positive BCH - algebra

Further

$(u_i \odot (u_i \odot b)) \odot b = (u_i \odot 0) \odot b = u_i \odot b = 0.$

means that (iv) is also satisfied

Theorem (3.3) provides some more extensions of BCK – algebra

The above theorem can be further extended as follows

Theorem (3.4)): - We recall E and E^1 of theorem (3.3).

We choose an object $c \notin E^1$ and let $E^* = E^1 \cup \{c\}$. We define a binary operation $\otimes$ on E^* as follows:

$u_i \otimes u_j = u_i * u_j = u_i$ (3.1.13)

for $i, j = 0, 1, 2, \ldots, n-1, i \neq j$

For $i = j$, $u_i \otimes u_j = u_i * u_i = 0$ (3.1.14)

$0 \otimes b = 0 \otimes c = 0$, $b \otimes b = 0 = c \otimes c$ (3.1.15)

$b \otimes 0 = b$, $c \otimes 0 = c$ (3.1.16)

$b \otimes c = b$, $c \otimes b = c$ (3.1.17)

We take two fixed suffixes k and l between 1 to n – 1 with $k \neq l$. Let

$u_k \otimes b = b$, $b \otimes u_k = 0$, $u_l \otimes c = c$, $c \otimes u_l = 0$ (3.1.18)

$u_i \otimes b = u_i$, $b \otimes u_i = b$ for $i = 1, 2, \ldots, k-1, k+1, \ldots, n-1$ (3.1.19)

and

$u_i \otimes c = u_i$, $c \otimes u_i = c$ for $i = 1, 2, \ldots, l-1, l+1, \ldots, n-1$ (3.1.20)

Then $(E^*, \otimes, 0)$ is a BCK – algebra.

Proof:

Conditions (i), (ii)and (iii) of BCK- algebra are satisfied from the above definitions.

Now we check conditions (iv) and (v) for some particular elements of E^*

We have

$(b \otimes (b \otimes c)) \otimes c = (b \otimes b) \otimes c = 0 \otimes c = 0.$

$(c \otimes (c \otimes b)) \otimes b = (c \otimes c) \otimes b = 0 \otimes b = 0.$

$(u_k \otimes (u_k \otimes c)) \otimes c = (u_k \otimes u_k) \otimes c = 0 \otimes c = 0.$

$(b \otimes (b \otimes u_l)) \otimes u_l = (b \otimes b) \odot u_l = 0 \odot u_l = 0$

For $i \neq k$, $(u_i \otimes (u_i \otimes c)) \otimes c = (u_i \otimes u_i) \otimes c = 0 \otimes c = 0$

For $i \neq l$, $(u_i \otimes (u_i \otimes b)) \otimes b = (u_i \otimes u_i) \otimes b = 0 \otimes b = 0$

Again, for $i \neq k$, $i \neq l$,

$((u_i \otimes b) \otimes (u_i \otimes c)) \otimes (c \otimes b) = (u_i \otimes u_i) \otimes c = 0 \otimes c = 0.$

Also, for $i = k$, $i \neq l$,

$((u_i \otimes b) \otimes (u_i \otimes c)) \otimes (c \otimes b) = (b \otimes u_i) \otimes c = 0 \otimes c = 0$

$((b \otimes u_i) \otimes (b \otimes c)) \otimes (c \otimes u_i) = (0 \otimes b) \otimes c = 0 \otimes c = 0$

Condition (v) can be proved for other elements of E^*.

Hence $(E^*, \otimes, 0)$ is a BCK – algebra

Theorem (3.5): - Let $E = \{0 \equiv u_0, u_1, u_2, \ldots, u_{n-1}\}$ be a finite set and let * be a binary operation in E satisfying conditions (3.1.1) and (3.1.2). Suppose that some of the pairs $\{u_i, u_j\}$ $i \neq j$ contains mutually disjoint elements (i.e., $u_i * u_j = u_i$ and $u_j * u_i = u_j$). Then binary operation '*' can be

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extended to other pairs such that $(E, *, u_0)$ becomes a BCK – algebra.

Proof: -

We divide pairs $\{i, j\}$, $i \neq j$, $i, j = 1,2,3, \ldots n-1$, into two classes A and B such that for $\{i, j\} \in A$, pairs $\{u_i, u_j\}$ contain mutually disjoint elements. For pairs $\{i, j\} \in B$ we extend '*' as follows:

$u_i * u_j = u_i$ and $u_j * u_i = 0$ for $i < j$ (3.1.21) (a)

(or, $u_i * u_j = 0$ and $u_j * u_i = u_j$ for $i < j$). (3.1.21) (b)

Conditions (i), (ii)and (iii) of BCK- algebra are satisfied from the given conditions. Now it remains to prove condition (iv) and (v).

For $\{i, j\} \in A$ we have

$(u_i * (u_i * u_j)) * u_j = (u_i * u_i) * u_j = 0 * u_j = 0,$

And $(u_j * (u_j * u_i)) * u_i = (u_j * u_j) * u_i = 0 * u_i = 0.$

For $\{i, j\} \in B$ and $i < j$ we have

$(u_i * (u_i * u_j))\ u_j = (u_i * u_i) * u_j = 0 * u_j = 0,$

$(u_j * (u_j * u_i)) * u_i = (u_j * 0) * u_i = u_j * u_i = 0.$

Again for $\{i, j\} \in B$ and $i > j$ we have

$(u_i * (u_i * u_j)) * u_j = (u_i * 0) * u_j = u_i * u_j = 0.$

So, condition (iv) is satisfied in all cases.

Let $i \neq j \neq k$ be three indexes between 1 to n – 1. Such that $\{i, j\} \in A$, $\{j, k\} \in B$, $\{i, k\} \in A$. Then

$((u_i * u_j) * (u_i * u_k)) * (u_k * u_j) = (u_i * u_i) * u_j = 0 * u_j = 0$

or $0 * 0 = 0,$

$((u_j * u_k) * (u_j * u_i)) * (u_i * u_k) = (u_j * u_j) * u_i = 0 * u_i = 0$

or $(0 * u_j) * u_i = 0 * u_i = 0,$

$((u_k * u_j) * (u_k * u_i)) * (u_i * u_j) = (0 * u_k) * u_i = 0 * u_i = 0$

or $(u_k * u_k) * u_i = 0 * u_i = 0,$

Thus, it can be proved that (v) is satisfied in all cases.

Hence $(E, *, u_0 = 0)$ is a BCK algebra.

As an illustration of theorem (3.5) we have the following

Example (3.1):

Let $E = \{a_0, a_1, a_2, \ldots, a_7\}$. Let a binary operation '*' be defined in E satisfying conditions (3.1.1) and (3.1.2). Further, suppose that pairs $\{a_1, a_2\}$, $\{a_2, a_3\}$, $\{a_3, a_4\}$, $\{a_4, a_5\}$, $\{a_5, a_6\}$, $\{a_6, a_7\}$, and $\{a_4, a_7\}$ contain mutually disjoint elements. Then binary operation '*' can be extended as given by the following table.

*	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_0	a_0	a_0	a_0	a_0	a_0	a_0	a_0	a_0
a_1	a_1	a_0	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_2	a_2	a_0	a_2	a_2	a_2	a_2	a_2
a_3	a_3	a_0	a_3	a_0	a_3	a_3	a_3	a_3
a_4	a_4	a_0	a_0	a_4	a_0	a_4	a_4	a_4
a_5	a_5	a_0	a_0	a_0	a_5	a_0	a_5	a_5
a_6	a_6	a_0	a_0	a_0	a_0	a_6	a_6	a_6
a_7	a_7	a_0	a_0	a_0	a_7	a_0	a_7	a_0

Table (3.3)

It can be verified that all conditions for BCK algebra are satisfied. Hence, table (3.3) represents a BCK algebra.

As an illustration of the remark (3.1) we consider the following example.

Example (3.2):

Let $X = \{0, a_1, a_2, a_3, a_4, a_5\}$ where elements of $\{0, a_1, a_2, a_3, a_4\}$ are mutually disjoint and the binary operation Θ is defined by the table

Θ	0	a_1	a_2	a_3	a_4	a_5
0	0	0	0	0	0	0

a_1	a_1	0	a_1	a_1	a_1	a_1
a_2	a_2	a_2	0	a_2	a_2	a_5
a_3	a_3	a_3	a_3	0	a_3	
a_4	a_4	a_4	a_4	a_4	0	a_4
a_5	a_5	a_5	0	a_5	a_5	0

Table (3.4)

Now we see that,

$((a_3 \Theta a_2) \Theta (a_3 \Theta a_5)) \Theta (a_5 \Theta a_2) = (a_3 \Theta 0) \Theta 0 = a_3 \neq 0$

$(a_3 \Theta a_2) \Theta a_5 = a_3 \Theta a_5 = 0$, $(a_3 \Theta a_5) \Theta a_2 = 0 \Theta a_2 = 0$

$(a_2 \Theta a_5) \Theta a_3 = a_5 \Theta a_3 = a_5$, $(a_2 \Theta a_3) \Theta a_5 = a_2 \Theta a_5 = a_5$

$(a_5 \Theta a_3) \Theta a_2 = a_5 \Theta a_2 = 0$, $(a_5 \Theta a_2) \Theta a_3 = 0 \Theta a_3 = 0$.

Also $(a_3 \Theta (a_3 \Theta a_5) \Theta a_5 = (a_3 \Theta 0) \Theta a_5 = a_3 \Theta a_5 = 0$.

This means that $(X, \Theta, 0)$ is a positive BCH – algebra but it is not a BCK – algebra because

$((a_3 * a_2) * (a_3 * a_5)) * (a_5 * a_2) = (a_3 * 0) * 0 = a_3 \neq 0$.

4. CONCLUSION

Form the above results we have proved some theorems on mutually Disjoint elements of a BCK-Algebra. We had extended the idea of mutually disjoint elements for any number of elements in the set of BCK-Algebra. The theorem has also verified with some specific examples. Our method provides extension of one BCK-Algebra of mutually disjoint element to another BCK-Algebra with some elements which are not mutually disjoint. This idea can also be used as extension of properties of BCK and BCH algebra in future.

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